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RELATIVISTIC CORRECTIONS TO DOPPLER MEASUREMENTS

Prepared by

L. M. Noble

SPACE TECHNOLOGY LABORATORIES, INC.
Redondo Beach, California

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Prepared: Larry M. Noble
Larry M. Noble
Theoretical Physics Department

Approved: Saul Altshuler
Saul Altshuler
Manager
Theoretical Physics Department

Approved: R. B. Muchmore
R. B. Muchmore
Director
Physical Research Division

QUANTUM PHYSICS LABORATORY
PHYSICAL RESEARCH DIVISION
SPACE TECHNOLOGY LABORATORIES, INC.
One Space Park
Redondo Beach, California

ABSTRACT

Relativistic corrections to rocket velocity determinations from Doppler frequency measurements are investigated for several configurations specified by the Guidance Laboratory. The results are correct to order $(\frac{v}{c})^2$ if gravitational and atmospheric effects can be neglected.

TABLE OF CONTENTS

	Page
I. INTRODUCTION	1
II. FREQUENCY TRANSFORMATIONS	1
III. MISSILE GUIDANCE CONFIGURATIONS	4
Configuration 1	4
Configuration 2	5
Configuration 3	5
Configuration 4	5
Configuration 5	6
Configuration 6	7
Configuration 7	7
IV. GENERAL REMARKS	10
REFERENCES	12

I. INTRODUCTION

The following is a brief discussion of Doppler frequencies correct to order $(\frac{v}{c})^2$ if gravitational influences are negligible. This discussion has been requested by the Guidance Laboratory¹ in connection with missile guidance problems. The problem has been discussed previously by H.C. Corben².

In the following section we define coordinates appropriate for studying the problem and write down the frequency transformations which are needed. In the next section this preliminary theory is applied in a straightforward way to several particular configurations of interest to the Guidance Laboratory¹. Finally, the significance of relativistic effects for guidance problems in general is reviewed, and a brief indication is given of the magnitude of the gravitational corrections.

II. FREQUENCY TRANSFORMATIONS

We consider an observer (presumably on earth) to be at rest at the origin of a coordinate system K and a missile at rest in a system K' which moves in the positive x direction with speed v relative to K. The line joining the missile and the observer is taken to be in the x-y plane, and the z-axis completes a right-hand coordinate system. The K' axes are taken to be parallel to the respective K axes, so that both systems are uniquely specified. (cf. Figure 1). The angle α is measured from the x-axis to the line directed from observer to missile as shown, i.e., $\vec{r} \cdot \vec{v} = |\vec{r}| v \cos \alpha$ where \vec{r} is the instantaneous

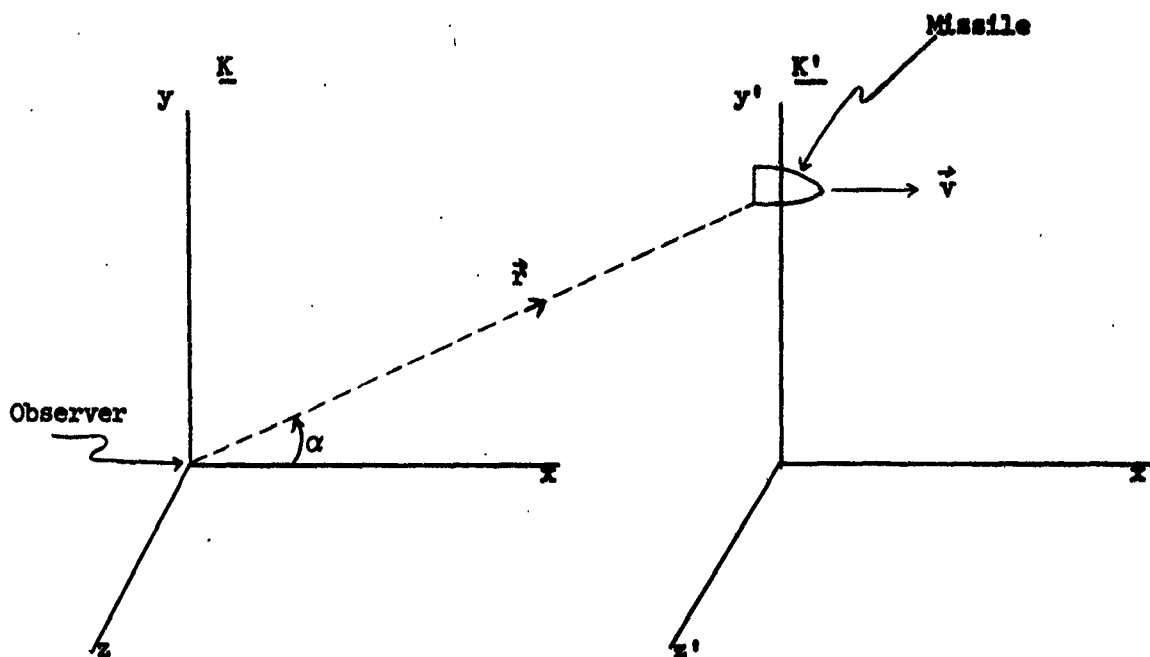


Figure 1

position vector of the missile relative to earth. (Note: This definition of angle differs from that used in (2) in that there the angle is measured from the x-axis to the propagation vector of radiation. So, for radiation from observer to missile our angle α and their θ_E are the same, but for radiation from missile to observer the two angles differ by π .) All unprimed quantities are measured relative to K; primed quantities relative to K'. Transformations from unprimed to primed coordinates are then given by the Lorentz transformations:³

$$x' = \frac{x - vt}{\sqrt{1 - \beta^2}}; y' = y; z' = z; t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \beta^2}}; \beta = \frac{v}{c}. \quad (1)$$

(Here it is presumed that the origins of K and K' coincide at $t = 0$).

Suppose an electromagnetic signal of frequency ν is transmitted from the observer on earth to the missile. The frequency transformation is determined by requiring that the phase of the wave at a given space-time point be an invariant, i.e.,

$$2\pi\nu \left(t - \frac{x \cos \alpha + y \sin \alpha}{c} \right) = 2\pi\nu' \left(t' - \frac{x' \cos \alpha' + y' \sin \alpha'}{c} \right). \quad (2)$$

Using (1) and equating coefficients of x , y , and t gives

$$\nu' = \nu \frac{1 - \beta \cos \alpha}{\sqrt{1 - \beta^2}}; \quad \cos \alpha' = \frac{\cos \alpha - \beta}{1 - \beta \cos \alpha}. \quad (3)$$

Radiation from missile to observer transforms in the same way except that now the appropriate angle is $\alpha + \pi$, so that

$$\nu' = \nu \frac{1 + \beta \cos \alpha}{\sqrt{1 - \beta^2}}; \quad \cos \alpha' = \frac{\cos \alpha + \beta}{1 + \beta \cos \alpha}. \quad (4)$$

In each case the angle α is measured relative to K at the instant the signal reaches (or leaves) the missile. If the velocity \vec{v} is not constant, the appropriate \vec{v} is also measured at that instant; then no corrections due to missile acceleration are necessary². We shall also require the time dilation transformation, which follows from (1). A time interval $(\Delta t)'$ measured at a fixed position in system K' is observed in system K as an interval (Δt) where

$$(\Delta t) = \frac{(\Delta t)'}{\sqrt{1 - \beta^2}}. \quad (5)$$

These equations are sufficient for studying the particular guidance problems posed by the Guidance Laboratory. (These results are, of course, independent of our choice of coordinates.)

III. MISSILE GUIDANCE CONFIGURATIONS

Seven particular guidance problems are defined in reference (1). We consider these individually.

Configuration 1

A signal of known frequency ν_1 is transmitted from the ground and received at the missile. (The transit time is simply given by $\frac{|\vec{r}_1|}{c}$; no transformation of this quantity to system K' is required since in practice all pertinent measurements are made on earth.) The missile receives a signal of frequency ν'_1 related to ν_1 by Eq. (3). A transponder in the missile has a known internal time delay $(\Delta t)'$ which is related to a time interval (Δt) measured on the ground by Eq. (5). After this delay, the transponder transmits a signal of frequency ν'_2 where

$$\nu'_2 = a \nu'_1 + b ; \quad (6)$$

the constants a and b are known in the rest frame of the missile. This signal is received on the ground with frequency ν_2 related to ν'_2 by Eq. (4). Thus, one has

$$\nu_2 = \left[a \frac{(1 - \beta_1 \cos \alpha_1)}{\sqrt{1 - \beta_1^2}} \nu_1 + b \right] \frac{\sqrt{1 - \beta_2^2}}{1 + \beta_2 \cos \alpha_2} . \quad (7)$$

Here β_1 and α_1 refer to the velocity and angle at the instant the first signal reaches the missile; β_2 and α_2 at the instant the second signal leaves the missile. Thus, if the first signal leaves the earth at $t = t_0$, then β_1 and α_1 are measured at $t = t_0 + \frac{|\vec{r}_1|}{c}$, and β_2 and α_2 are measured at $t = t_0 + \frac{|\vec{r}_1|}{c} + (\Delta t)$. In general α_1 and α_2 will

differ since the missile moves a distance $\vec{v}(\Delta t)$ during the time between measurements. If the missile velocity is constant, however, $\beta_1 = \beta_2$, and Eq. (7) becomes

$$\nu_2 = a \frac{(1 - \beta \cos \alpha_1)}{(1 + \beta \cos \alpha_2)} \nu_1 + b \frac{\sqrt{1 - \beta^2}}{(1 + \beta \cos \alpha_2)} \quad (8)$$

Configuration 2

The system here is identical to that in configuration 1 except that here the ground receiver is some distance away from the transmitter. If we define α_1 by the relation $\vec{v}_1 \cdot \vec{r}_1 = v_1 |\vec{r}_1| \cos \alpha_1$, measuring \vec{r}_1 from the ground transmitter at $t = t_0 + \frac{|\vec{r}_1|}{c}$ as before, and α_2 by $\vec{v}_2 \cdot \vec{r}_2 = v_2 |\vec{r}_2| \cos \alpha_2$ where \vec{r}_2 is measured from the ground receiver at $t = t_0 + \frac{|\vec{r}_1|}{c} + (\Delta t)$, then Eq. (7) and, for $\vec{v} = \text{const.}$, Eq. (8) apply to this geometry also. The two situations are precisely the same except that now α_1 and α_2 are measured from different ground positions as well as at different times.

Configuration 3

Here the missile simply transmits a signal of known frequency ν' which is received on the ground with frequency ν given by Eq. (4). Velocity and angle are measured when the signal leaves the missile.

Configuration 4

In this case the missile transmits a signal of known frequency ν_1' which is received on the ground with frequency ν_1 given by Eq. (4). (Here again α and \vec{v} are measured when the signal leaves the missile; the transit time is $\frac{|\vec{r}_1|}{c}$ as usual.) After a known time delay (Δt) a ground

transponder sends a signal of frequency

$$\nu_2 = a\nu_1 + b \quad (9)$$

The missile receives this signal with frequency ν_2' given by Eq. (3); this frequency is measured in the missile and the information transmitted digitally to earth. Thus,

$$\nu_2' = \left[a \frac{\sqrt{1 - \beta_1^2}}{1 + \beta_1 \cos \alpha_1} \nu_1' + b \right] \frac{1 - \beta_2 \cos \alpha_2}{\sqrt{1 - \beta_2^2}} \quad (10)$$

As usual, the subscript 1 refers to measurements made when the first signal leaves the missile; 2 to measurements made when the second signal reaches the missile. For instance, let the first signal reach the ground at $t = t_0$. Thus, α_1 and v_1 are measured at $t = t_0 - \frac{|\vec{r}_1|}{c}$, and α_2 and v_2 are measured at $t = t_0 + (\Delta t) + \frac{|\vec{r}_2|}{c}$. For the special case of constant velocity one has $\beta_1 = \beta_2$, so that Eq. (1) simplifies as before to

$$\nu_2' = a \frac{(1 - \beta \cos \alpha_2)}{(1 + \beta \cos \alpha_1)} \nu_1' + b \frac{(1 - \beta \cos \alpha_2)}{\sqrt{1 - \beta^2}} \quad (11)$$

Configuration 5

Here the configuration is the same as configuration 4 except that the second signal (with frequency ν_2') is re-radiated from the missile and received on the ground with frequency ν_3 related to ν_2' by Eq. (4). If we assume that no time delay is involved in this re-radiation process, then the angle and velocity appearing in this last equation are just α_2 and β_2 , so that one obtains finally

$$\nu_3 = \left[a \frac{\sqrt{1 - \beta_1^2}}{1 + \beta_1 \cos \alpha_1} \nu'_1 + b \right] \frac{(1 - \beta_2 \cos \alpha_2)}{(1 + \beta_2 \cos \alpha_2)} . \quad (12)$$

Subscripts refer to measurements at the same times as in configuration 4.

Configuration 6

Here configurations 4 and 5 are repeated with several transponders at different ground positions. The equations already given then apply exactly for each transponder provided the value of α appropriate to that transponder is used in each case.

Configuration 7

Now consider the problem in which two missiles move relative to the earth and to each other. Signals can be transmitted between the two missiles and interpreted in terms of Doppler frequencies. In this case all of the previous results are still applicable; however, now the appropriate velocities and angles are measured in the rest frame of one of the missiles. We consider the following specific geometry. Relative to system K (the earth) let missile number 1 have position \vec{r}_1 at time t and let missile number 2 have position \vec{r}_2 at time t . Let system K' be the rest frame of missile number 1, and assume $\vec{v}_1 = \text{constant}$; then relative to K' missile number 2 has position \vec{r}'_2 at time t' with transformations given by Eqs. (1). (See Figure 2). Let missile number 2 have velocity \vec{v}_2 at time t . Then its velocity relative to missile number 1 is obtained by differentiating Eqs. (1).

$$\frac{dx'_2}{dt'} = \frac{\frac{dx_2}{dt} - v_1}{1 - \frac{v_1}{c} \frac{dx_2}{dt}} ; \quad \frac{dy'_2}{dt'} = \frac{\frac{dy_2}{dt} \sqrt{1 - \beta_1^2}}{1 - \frac{v_1}{c} \frac{dx_2}{dt}} ; \quad \frac{dz'_2}{dt'} = \frac{\frac{dz_2}{dt} \sqrt{1 - \beta_1^2}}{1 - \frac{v_1}{c} \frac{dx_2}{dt}} . \quad (13)$$

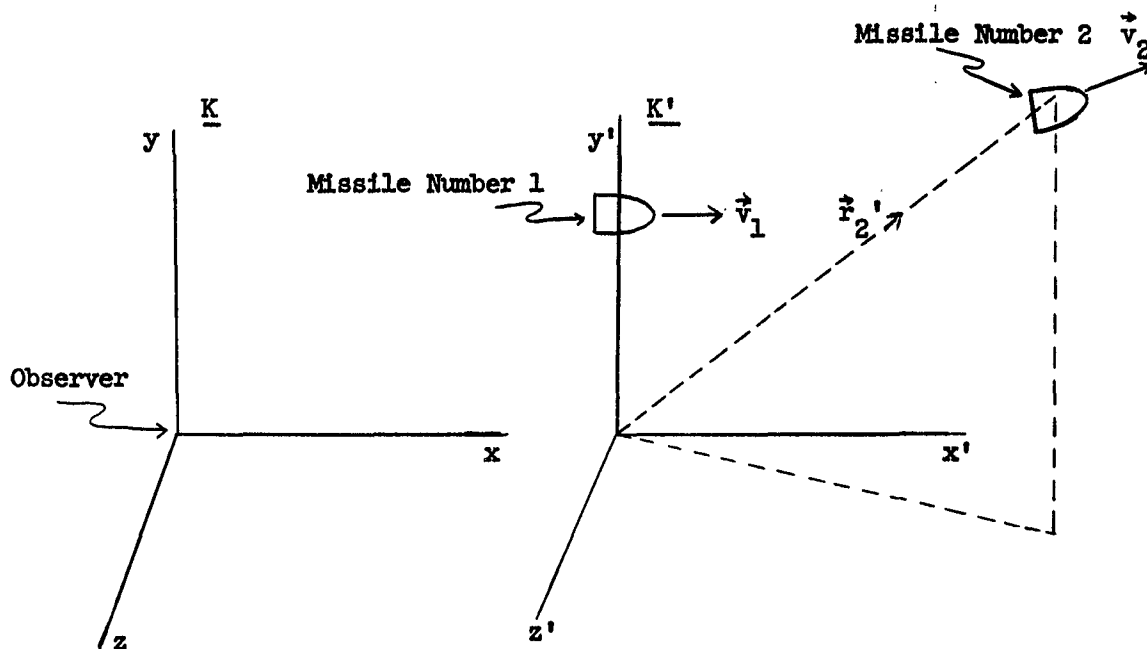


Figure 2

These equations express the velocity components of missile number 2 relative to missile number 1 in terms of its velocity relative to earth. Doppler frequencies of signals between the two missiles then depend on v'_2 where

$$(v'_2)^2 = \left(\frac{dx'_2}{dt'}\right)^2 + \left(\frac{dy'_2}{dt'}\right)^2 + \left(\frac{dz'_2}{dt'}\right)^2 \quad (1)$$

$$= \frac{1}{\left[1 - \frac{v_1}{c^2} \frac{dx_2}{dt}\right]^2} \left\{ \left(\frac{dx_2}{dt} - v_1\right)^2 + (1 - \beta_1^2) \left[\left(\frac{dy_2}{dt}\right)^2 + \left(\frac{dz_2}{dt}\right)^2\right] \right\},$$

and upon $\cos \gamma'_2$ given by

$$\cos \gamma'_2 = \frac{(\vec{r}'_2 - \vec{r}'_1) \cdot \vec{v}'_2}{|\vec{r}'_2 - \vec{r}'_1| v'_2} \quad (1)$$

Let

$$R_2'^2 \equiv |\vec{r}_2' - \vec{r}_1'|^2 = (x_2')^2 + (y_2' - y_1')^2 + (z_2')^2$$

$$= \frac{(x_2 - v_1 t)^2}{(1 - \beta_1^2)} + (y_2 - y_1)^2 + z_2^2 \quad (16)$$

Then

$$\cos \gamma_2' = \frac{1}{R_2' v_2'} \left[x_2' \frac{dx_2'}{dt'} + (y_2' - y_1') \frac{dy_2'}{dt'} + z_2' \frac{dz_2'}{dt'} \right]$$

$$= \frac{1}{R_2' v_2' (1 - \frac{v_1}{c} \frac{dx_2}{dt})} \left\{ \frac{(x_2 - v_1 t)}{\sqrt{1 - \beta_1^2}} \left(\frac{dx_2}{dt} - v_1 \right) \right. \quad (17)$$

$$\left. + \left[(y_2 - y_1) \frac{dy_2}{dt} + z_2 \frac{dz_2}{dt} \right] \sqrt{1 - \beta_1^2} \right\}.$$

Consider, for example, the two-missile analogue of configuration 3.

Let missile number 2 transmit a signal of known frequency γ'' at time t .

As measured on earth, the signal arrives at missile number 1 at time $t + (\Delta t)$ where $(\Delta t) = \frac{|\vec{r}_2 - \vec{r}_1 - \vec{v}_1(\Delta t)|}{c}$. (Here \vec{r}_1 and \vec{r}_2 are measured at time t .) The frequency measured in missile number 1 is then γ' given by Eq. (4);

$$\gamma' = \gamma'' \frac{\sqrt{1 - \frac{v_2'^2}{c^2}}}{1 + \frac{v_2'}{c} \cos \gamma_2'} \quad (18)$$

with v_2' and $\cos \gamma_2'$ as given above. This procedure is applicable to any configuration provided one uses the velocity and angular measurements for one missile relative to the other as given by Eqs. (14) and (17). (The restriction $\vec{v}_1 = \text{constant}$ can be removed by measuring \vec{v}_1 at the correct times.)

IV. GENERAL REMARKS

Doppler frequencies correct to order $(\frac{v}{c})^2$ have been given for several missile guidance configurations, and an extension of these derivations to more complex configurations is straightforward. The general question of whether or not relativistic corrections are important depends, of course, on the magnitude of the second-order term as compared with the linear term. For the case of ballistic missiles, velocities on the order of 5×10^4 ft/sec are of interest, corresponding to $\beta \approx 5 \times 10^{-5}$. Neglect of relativistic effects, therefore, introduces an error of nearly one part in 10^4 , which may be significant. For space vehicles even larger velocities may be involved.

The neglect of gravitational influences is completely justified. Radiation from the earth's surface to a missile at an altitude H experiences a frequency decrease $\Delta\nu$ given by

$$\frac{\Delta\nu}{\nu} = \frac{\phi(r_0 + H) - \phi(r_0)}{c^2}, \quad (19)$$

where $\phi(r)$ is the gravitational potential and r_0 is the radius of the earth. (This phenomenon is the well-known "red shift" often discussed with regard to the general theory of relativity. The result can most easily be derived approximately by considering a photon of mass $m = \frac{h\nu}{c^2}$ in the earth's gravitational field.) For the case $H = 1000$ miles, Eq. (19) gives $\frac{\Delta\nu}{\nu}$ of order -10^{-10} , which is completely negligible. Furthermore, under two-way transmission the effect tends to cancel out.

Finally, it should be noted that these results apply to the ideal case of transmission through a vacuum. Deviations from this ideal situation, due, for example, to the earth's atmosphere, must be accounted for separately by standard techniques.

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